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ARTICLE

Numerical Solution of Optimal Feedback Control for Finite Horizon Optimal Control Problem with Growth Condition

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Abstract

In this paper, we study approximation of optimal feedback control for the finite horizon optimal control problem with growth conditions for control systems and cost functionals. The optimal control problems with growth conditions are usually applied in practice. In reducing the boundedness for the dynamics function to the growth conditions, we have some difficulty in proving the convergence of the approximation scheme by the Euler scheme. First, we prove comparison principle of unbounded viscosity solution under the growth condition for the dynamics function and prove that the value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation. We then construct a new approximation scheme using Euler implicit scheme for the finite horizon optimal control problem and prove that the approximation value function converges to the viscosity solution of the HJB equation as the discretization step goes to zero. Based on the approximations of the viscosity solution, we construct the optimal feedback controls for the corresponding discrete control systems. We consider an example to show the efficiency of the time-space approximation algorithm by the Euler implicit scheme through computational results.

Keywords: Optimal feedback control, Hamilton-Jacobi-Bellman equation, viscosity solution, uniqueness, approximation scheme, convergence

1. INTRODUCTION

The construction of optimal feedback control for an optimal control problem is a major issue in optimal control theory. It has been already proved that the value function is the unique viscosity solution of the HJB equation [1-5] and optimal feedback control can be obtained by approximating the viscosity solution. Therefore, Bellman's dynamic programming principle has now become a focus of modern control theory. Some researchers studied the convergence of approximation schemes for the viscosity solution of the HJB equation [6-13].

More recent results on the construction of an optimal feedback control and the convergence of the approximation schemes of the viscosity solution of the HJB equation for finite horizon optimal control problems can be found in Alla et al. [14], Falcone and Ferretti [15], Guo and Wu [16,17], Guo and Sun [18], Sun and Guo [19], and Kim and Hong [5]. Guo and Wu [16,17] constructed an optimal feedback control for the optimal control problem based on the approximations of the viscosity solution. We refer to [9,20,21] for the study of uniqueness and convergence of the approximation scheme of the viscosity solution of the Hamilton-Jacobi quasi-variational inequalities.

Barles et al. [22], Ramaswamy and Dharmatti [23], and Kim and Hong [5,24] studied the uniqueness of unbounded viscosity solutions and convergence of the approximation scheme for optimal control problems with unbounded cost functionals. Recently, Carrere and Zidani [25], Ramaswamy and Dharmatti [23], Sun and Guo [19], and Kim and Hong [5] considered the optimal control problem by reducing the boundedness on the dynamic function to the linear growth condition. In general, previous works for optimal control problems assumed the boundedness or the linear growth condition on the dynamics function in proving the convergence of the approximation scheme of the viscosity solution of the HJB equation.

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In this paper, we consider a finite horizon optimal control problem in which the dynamics function and the cost functional satisfy the growth conditions. The aim of this paper is to establish the uniqueness of the viscosity solution of the HJB equation and the convergence of the approximation scheme to the viscosity solution by reducing the boundedness conditions for the dynamics function and the cost functional into the growth conditions and to construct an optimal feedback control for the optimal control problem.

The rest of the paper is organized as follows. In Section 2 is proved the comparison principle of unbounded viscosity solutions of the HJB equation by converting the unbounded function into bounded one by suitable exponential transformation when the dynamics function and the cost functional satisfy the growth conditions.

In Section 3 is constructed a time discretization scheme by Euler implicit scheme for HJB equation and is proved that the approximation value function satisfies the growth condition and the local Lipschitz continuity. We then prove that the solutions of the time discretization scheme and time-space discretization scheme converge to the viscosity solution of the HJB equation when the discretization step goes to zero. Section 4 is devoted to construct the optimal feedback controls for the corresponding control systems based on the approximations of the viscosity solution. In Section 5, we present an algorithm for finding the viscosity solution an optimal feedback control algorithm based on the approximation of the time-space discretization scheme and give a numerical example.

2. UNIQUENESS OF VISCOSITY SOLUTIONS OF HJB EQUATION

Given $T > 0$, consider the following finite time optimal control problem.

$$\begin{cases} \dot{y}(\tau) = f\big(\tau, y, (\tau)u(\tau)\big), t < \tau \le T \\ y(t) = x \\ u(\tau) \in U \\ J\big(t, x, u(\cdot)\big) = \int_t^T \ell\left(\tau, y, (\tau)u(\tau)\right)d\tau + g\big(y(T)\big) \to \min_{u(\cdot)} \end{cases} \tag{2.1}$$

where $u(\cdot) \in \mathcal{U} = \{u(\cdot): [0,T] \to U \mid u\text{:measurable}\}$

We make the following assumptions for the finite time optimal control problem.

(**A**1). $f: [0,T] \times R^n \times U \to R^n$is continuous and satisfies the following conditions.

$$\forall R > 0, \exists L_f(R) > 0, \forall t, s \in [0,T], \forall x, y \in \bar{B}(0,R), \forall u \in U,$$
$$\| f(t,x,u) - f(s,y,u) \| \le L_f(R)(\mid t - s \mid + \| x - y \|).$$

(**A**2). $f: [0,T] \times R^n \times U \to R^n$satisfies the following condition.

$$\exists C > 0, \forall t \in [0,T], \forall x \in R^n, \forall u \in U, (f,(t,x,u)x) \le C(1+\| x \|^2).$$

(**A**3). $f: [0,T] \times R^n \times U \to R^n$satisfies the growth condition:

$$\forall t \in [0,T], \forall x \in R^n, \forall u \in U, \| f(t,x,u) \| \le C(1+\| x \|^p),$$

where $p \ge 1$is a constant and $\|\cdot\|$denotes the Euclidean norm of R^n.

(**A**4). $\ell: [0,T] \times R^n \times U \to R$ and $g: R^n \to R$ are continuous functions and satisfy the following conditions, respectively.

$$\forall R > 0, \exists L_\ell(R) > 0, \forall t, s \in [0,T], \forall x, y \in \bar{B}(0,R), \forall u \in U,$$
$$\mid \ell(t,x,u) - \ell(t,y,u) \mid \le L_\ell(R)(\mid t - s \mid + \| x - y \|),$$
$$\forall R > 0, \exists L_g(R) > 0, \forall x, y \in \bar{B}(0,R), \forall u \in U,$$
$$\mid g(x) - g(y) \mid \le L_g(R) \| x - y \|,$$
$$\exists C > 0, \forall t \in [0,T], \forall x \in R^n, \forall u \in U, \mid \ell(t,x,u) \mid \le C(1+\| x \|^2),$$
$$\exists C > 0, \forall x \in R^n, \mid g(x) \mid \le C(1+\| x \|^2).$$

The value function of the problem (2.1) is defined as

$$v(t,x) = \inf_{u \in \mathcal{U}} J(t,x,u).$$

Theorem 2.1. *Suppose that the assumptions (A1)-(A4) hold. Then, for value function* $v(t,x)$ *we have*

$$\exists C_1 > 0, \forall t \in [0,T], \forall x \in R^n, \mid v(t,x) \mid \leq C_1(1+\parallel x \parallel^2). \tag{2.2}$$

Proof. For $u(\cdot) \in \mathcal{U}$, we have

$$\mid v(t,x) \mid \leq \int_t^T \mid \ell\big(\tau, y, (\tau)u(\tau)\big) \mid d\tau + \mid g\big(y(T)\big) \mid \leq$$

$$\int_t^T C\,(1+\parallel y(\tau) \parallel^2)d\tau + C(1+\parallel y(T) \parallel^2). \tag{2.3}$$

On the other hand, from (A2), we have

$$\big(\dot{y}, (\tau)y(\tau)\big) = \Big(f, \big(\tau, y, (\tau)u(\tau)\big)y(\tau)\Big), t < \tau \leq T.$$

Hence, we have

$$\frac{1}{2}\frac{d}{d\tau}\parallel y(\tau) \parallel^2 = \Big(f, \big(\tau, y, (\tau)u(\tau)\big)y(\tau)\Big) \leq C(1+\parallel y(\tau) \parallel^2), t < \tau \leq T. \tag{2.4}$$

By integrating the both sides of (2.4), we have

$$\frac{1}{2}\parallel y(\tau) \parallel^2 \leq \frac{1}{2}\parallel x \parallel^2 + \int_t^\tau C\,(1+\parallel y(s) \parallel^2)ds$$

and from Gronwall-inequality, we obtain

$$\parallel y(\tau) \parallel^2 \leq \big(\parallel x \parallel^2 + C(\tau - t)\big)e^{C(\tau - t)} \leq (\parallel x \parallel^2 + 2CT)e^{2CT}, \tau \in (t,T].$$

It follows from (2.3) that

$$\mid v(t,x) \mid \leq \int_t^T C\,(1 + (\parallel x \parallel^2 + 2CT)e^{2CT})d\tau + C(1 + (\parallel x \parallel^2 + 2CT)e^{2CT})$$

$$\leq C(1+T)(1 + (\parallel x \parallel^2 + 2CT)e^{2CT}) \leq C_1(1+\parallel x \parallel^2). \tag{2.4}$$

By Theorem 2.1, the value function $v(t,x)$ is locally bounded, and for any $\alpha > 0$ and $t \in [0,T]$ we have $\lim_{\|x\| \to \infty} v(t,x)e^{-\alpha\xi(x)} = 0$, where $\xi(x) = (1+\parallel x \parallel^2)^{\frac{1}{2}}$.

Now, for any $t \in [0,T]$ and $\alpha > 0$, let $E([0,T] \times \mathrm{R}^n)$ denote the space of locally bounded functions u such that $\lim_{\|x\| \to \infty} u(t,x)e^{-\alpha\xi(x)} = 0$.

For a locally bounded function u, we have

$$u^*(t,x) = \limsup_{r \to 0}\{u, (s,y), : (s,y) \in [0,T] \times R^n \mid t - s \mid \leq r \parallel x - y \parallel \leq r\}$$

and

$$u_*(t,x) = \liminf_{r \to 0}\{u, (s,y), : (s,y) \in [0,T] \times R^n \mid t - s \mid \leq r \parallel x - y \parallel \leq r\}.$$

If $u = u^*$, u is upper semi-continuous function and if $u = u_*$, u is lower semi-continuous function.

Definition 2.2. Let u be a locally bounded function. Suppose that for any function $\psi \in C^1(R^{n+1})$ u satisfies the following inequality at a local minimum point $(t,x) \in (0,T) \times R^n$ of $u_* - \psi$ (local maximum point of $u^* - \psi$)

$$-\frac{\partial\psi(t,x)}{\partial t} + H\big(t, x, D_x\psi(t,x)\big) \geq 0.$$

$$\left(-\frac{\partial\psi(t,x)}{\partial t} + H\big(t, x, D_x\psi(t,x)\big) \leq 0.\right)$$

Then, u is called a viscosity super-solution (sub-solution) of the following HJB equation.

$$-\frac{\partial u(t,x)}{\partial t} + H\big(t, x, D_x u(t,x)\big) = 0. \tag{2.5}$$

If u is both a viscosity super-solution and sub-solution of (2.5), we say that u is a viscosity solution of (2.5). Also, if u is a viscosity super-solution (sub-solution) of (2.5) and satisfies the inequality

$$u_*(T,x) \geq g_*(x)\ \big(u^*(T,x) \leq g^*(x)\big),$$

we say that $u$is a viscosity super-solution (sub-solution) of HJB equation (2.5) satisfying the terminal condition

$$u(T,x) = g(x). \tag{2.6}$$

If u is both a viscosity super-solution and sub-solution of (2.5) and (2.6), we say that $u$is a viscosity solution of (2.5) and (2.6).

The value function $v(t,x)$of the optimal control problem (2.1) is the viscosity solution of the terminal problem (2.5) and (2.6) ([1]). Then, we have

$$H\big(t,x,D_x u(t,x)\big) = \sup_{u\in U}\{-f(t,x,u)\cdot D_x u(t,x) - \ell(t,x,u)\}.$$

Theorem 2.3. *Suppose that the assumptions (A1)-(A4) hold. If upper semicontinuous function* $v_1 \in E([0,T]\times R^n)$*is a viscosity sub-solution of (2.5) and lower semicontinuous function* $v_2 \in E([0,T]\times R^n)$*is a viscosity super-solution of (2.5) and*

$$v_1(T,x) \le v_2(T,x), x \in R^n,$$

we have

$$v_1(t,x) \le v_2(t,x), (t,x) \in [0,T]\times R^n.$$

Proof. For any fixed $\alpha > 0$, define $u_1$and $u_2$by

$$u_1(t,x) = v_1(t,x)e^{-\alpha\xi(x)} \text{ and } u_2(t,x) = v_2(t,x)e^{-\alpha\xi(x)},$$

respectively. Since $v_1$and $v_2 \in E([0,T]\times R^n)$, $u_1$and $u_2$are bounded upper semicontinuous function and bounded lower semicontinuous function, respectively.

First, let us prove that $u_2$is a viscosity super-solution of the following equation

$$-\frac{\partial u(t,x)}{\partial t} + \bar{H}\big(t,x,D_x u(t,x)\big) = 0, \tag{2.7}$$

where

$$\bar{H}\big(t,x,D_x u(t,x)\big) = \sup_{u\in U}\big\{-f(t,x,u)\cdot\big(\alpha u(t,x)D\xi(x) + D_x u(t,x)\big) - e^{-\alpha\xi(x)}\ell(t,x,u)\big\}.$$

For any $\psi \in C^1(R^{n+1})$, let (s,z)be a local minimum point of $u_2(t,x) - \psi(t,x)$. Then, there exists $r > 0$such that for any $(t,x) \in B\big((s,z),r\big)\cap(0,T)\times R^n$we have

$$u_2(t,x) - \psi(t,x) \ge u_2(s,z) - \psi(s,z).$$
$$u_2(t,x) - \phi(t,x) \ge u_2(s,z) - \phi(s,z) = 0.$$

From

$$e^{\alpha\xi(x)}\big(u_2(t,x) - \phi(t,x)\big) \ge e^{\alpha\xi(z)}\big(u_2(s,z) - \phi(s,z)\big) = 0,$$

it follows that for any $(t,x) \in B\big((s,z),r\big)\cap(0,T)\times R^n$

$$v_2(t,x) - e^{\alpha\xi(x)}\phi(t,x) \ge v_2(s,z) - e^{\alpha\xi(z)}\phi(s,z) = 0.$$

Therefore, if we choose $\Phi(t,x) = e^{\alpha\xi(x)}\phi(t,x)$, then $(s,z)$is the local minimum point of $v_2 - \Phi$. Since $v_2(t,x)$is a viscosity super-solution of the HJB equation (2.5), we have

$$-\frac{\partial\Phi(s,z)}{\partial t} + \sup_{u\in U}\{-f(s,z,u)\cdot D_z\Phi(s,z) - \ell(s,z,u)\} =$$

$$= -e^{\alpha\xi(z)}\frac{\partial\phi(s,z)}{\partial t} + \sup_{u\in U}\Big\{-f(s,z,u)\cdot\Big(\alpha D\xi(z)e^{\alpha\xi(z)}\phi(s,z) + e^{\alpha\xi(z)}D_x\phi(s,z)\Big) - \ell(s,z,u)\Big\} \ge 0,$$

which implies that

$$-\frac{\partial\phi(s,z)}{\partial t} + \sup_{u\in U}\big\{-f(s,z,u)\cdot\big(\alpha D\xi(z)\phi(s,z) + D_z\phi(s,z)\big) - e^{-\alpha\xi(z)}\ell(s,z,u)\big\} \ge 0,$$

and then $u_2(t,x)$is a viscosity super-solution of the HJB equation (2.7).

Similarly, it is proved that if $v_1(t,x)$is a viscosity sub-solution of the HJB equation (2.5), $u_1(t,x) = v_1(t,x)e^{-\alpha\xi(x)}$is a viscosity sub-solution of the HJB equation (2.7).

Next, for any $\alpha > 0$let us prove that $u_1(t,x) \le u_2(t,x)$, $(t,x) \in [0,T] \times R^n$. By contradiction, suppose that there exists an $\alpha > 0$and $(\bar{t},\bar{x}) \in [0,T] \times R^n$such that

$$u_1(\bar{t},\bar{x}) - u_2(\bar{t},\bar{x}) = \delta > 0.$$

Now, for any $\epsilon > 0$, let us define the following auxiliary function.

$$\Phi_\epsilon(t,s,x,y) = u_1(t,x) - u_2(s,y) - \frac{\| x-y \|^2 + | t-s |^2}{2\epsilon} - \beta(\langle x\rangle^m + \langle y\rangle^m) - \gamma\left(\frac{1}{t}+\frac{1}{s}\right) - \eta(2T-t-s),$$

where $\beta, m, \gamma, \eta > 0$and $\langle x\rangle = (1+\| x \|^2)^{\frac{1}{2}}$.

We see that $\Phi_\epsilon$is upper semi-continuous function. If $\| x \|$and $\| y \|$are large enough, then $\Phi_\epsilon \le 0$and so there exists a supremum of Φ_ϵ. Then,

$$\sup_{[0,T]^2 \times R^n \times R^n} \Phi_\epsilon(t,s,x,y) = \Phi_\epsilon(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon).$$

Now, if there exists $m_0, \beta_0, \gamma_0, \eta_0 > 0$such that

$$\forall m \le m_0, \forall \beta \le \beta_0, \forall \gamma \le \gamma_0, \forall \eta \le \eta_0, 2\beta\langle \bar{x}\rangle^m - 2\gamma\frac{1}{\bar{t}} - 2\eta(T-\bar{t}) \le \frac{\delta}{2},$$

it follows that

$$\frac{\delta}{2} \le \delta - 2\beta\langle \bar{x}\rangle^m - 2\gamma\frac{1}{\bar{t}} - 2\eta(T-\bar{t}) = \Phi_\epsilon(\bar{t},\bar{t},\bar{x},\bar{x}) \le \Phi_\epsilon(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon).$$

Therefore, we obtain

$$\frac{| t_\epsilon - s_\epsilon |^2 + \| x_\epsilon - y_\epsilon \|^2}{2\epsilon} \le u_1(t_\epsilon, x_\epsilon) - u_2(s_\epsilon, y_\epsilon).$$

From the boundedness of $u_1$and u_2, it follows that

$$| t_\epsilon - s_\epsilon | \to 0, \| x_\epsilon - y_\epsilon \| \to 0 \ (\epsilon \to 0). \tag{2.8}$$

Also, for any $m \le m_0$and $\beta \le \beta_0$, since

$$\beta(\langle x_\epsilon\rangle^m + \langle y_\epsilon\rangle^m) \le \| u_1 \|_\infty + \| u_2 \|_\infty = L,$$

we have $x_\epsilon, y_\epsilon \in \bar{B}\left(0, (L/\beta)^{\frac{1}{m}}\right)$, and since

$$\gamma\left(\frac{1}{t_\epsilon}+\frac{1}{s_\epsilon}\right) \le L,$$

if we choose $\gamma_1 = \min\{\gamma_0, LT\}$, for any $\gamma \le \gamma_1$we have $t_\epsilon, s_\epsilon \in [\gamma, /LT]$. Thus, $(t_\epsilon, x_\epsilon), (s_\epsilon, y_\epsilon) \in K := [\gamma, /LT] \times \bar{B}\left(0, (L/\beta_0)^{\frac{1}{m_0}}\right)$. Suppose that

$$s := \sup_{[0,T]\times K}\left\{(u_1-u_2)(t,x) - 2\beta\langle x\rangle^m - 2\eta(T-t) - \frac{2\gamma}{t}\right\}.$$

Then

$$s \le \Phi_\epsilon(t_\epsilon, s_\epsilon, x_\epsilon, y_\epsilon) \le u_1(t_\epsilon, x_\epsilon) - u_2(s_\epsilon, y_\epsilon) - \beta(\langle x_\epsilon\rangle^m + \langle y_\epsilon\rangle^m) - \gamma\left(\frac{1}{t_\epsilon}+\frac{1}{s_\epsilon}\right) - \eta(2T - t_\epsilon - s_\epsilon) = s(\epsilon)$$

We show that $s(\epsilon) \to s$as $\epsilon \to 0$. From (2.8) and the compactness of K, without loss of generality, we can assume that $t_\epsilon \to t^*, s_\epsilon \to t^*, x_\epsilon \to x^*, y_\epsilon \to x^*$as $\epsilon \to 0$. By the compactness of $K$and the upper semi-continuity of $s(\epsilon)$, we have

$$\limsup_{\epsilon\to 0} s(\epsilon) = u_1(t^*,x^*) - u_2(t^*,x^*) - 2\beta\langle x^*\rangle^m - 2\eta(T-t^*) - 2\gamma\frac{1}{t^*} \le s.$$

Hence, as $\epsilon \to 0$, we have

$$\frac{\| x_\epsilon - y_\epsilon \|^2}{2\epsilon} \to 0, \frac{| t_\epsilon - s_\epsilon |^2}{2\epsilon} \to 0. \tag{2.9}$$

Now, consider the case of $t_\epsilon = T$. Then, $t^* = T$ and

$$\limsup_{\epsilon\to 0}\Phi_\epsilon(t_\epsilon,s_\epsilon,x_\epsilon,y_\epsilon)\le\limsup_{\epsilon\to 0}\big(u_1(t_\epsilon,x_\epsilon)-u_2(s_\epsilon,y_\epsilon)\big)=$$
$$=u_1(t^*,x^*)-u_2(t^*,x^*)=u_1(T,x^*)-u_2(T,x^*)\le 0,$$

which is contradictory to $\limsup_{\epsilon\to 0}\Phi_\epsilon(t_\epsilon,s_\epsilon,x_\epsilon,y_\epsilon)>0$. The case of $s_\epsilon=T$is proved similarly.

Next, we consider the case of $(t_\epsilon,x_\epsilon),(s_\epsilon,y_\epsilon)\in(0,T)\times R^n$. Define the following auxiliary functions:

$$\phi(t,x)=u_2(s_\epsilon,y_\epsilon)+\frac{\|\,x-y_\epsilon\,\|^2+|\,t-s_\epsilon\,|^2}{2\epsilon}+\beta(\langle x\rangle^m+\langle y_\epsilon\rangle^m)+$$
$$\gamma\left(\frac{1}{t}+\frac{1}{s_\epsilon}\right)+\eta(2T-t-s_\epsilon)$$

and

$$\psi(s,y)=u_1(t_\epsilon,x_\epsilon)-\frac{\|\,x_\epsilon-y\,\|^2+|\,t_\epsilon-s\,|^2}{2\epsilon}-\beta(\langle x_\epsilon\rangle^m+\langle y\rangle^m)-$$
$$\gamma\left(\frac{1}{t_\epsilon}+\frac{1}{s}\right)-\eta(2T-t_\epsilon-s).$$

Then $u_1-\varphi$has maximum at $(t_\epsilon,x_\epsilon)$and $u_2-\psi$has minimum at (s_ϵ,y_ϵ).

From the definition of the viscosity solution, we have

$$-\frac{\partial\varphi(t_\epsilon,x_\epsilon)}{\partial t}+\bar H\big(t_\epsilon,x_\epsilon,D_x\varphi(t_\epsilon,x_\epsilon)\big)\le 0$$

at $(t_\epsilon,x_\epsilon)$and

$$-\frac{\partial\psi(s_\epsilon,y_\epsilon)}{\partial s}+\bar H\left(s_\epsilon,y_\epsilon,D_y\psi(s_\epsilon,y_\epsilon)\right)\ge 0$$

at (s_ϵ,y_ϵ). Thus we have

$$\psi_s(s_\epsilon,y_\epsilon)-\varphi_t(t_\epsilon,x_\epsilon)\le\bar H\left(s_\epsilon,y_\epsilon,D_y\psi(s_\epsilon,y_\epsilon)\right)-\bar H\big(t_\epsilon,x_\epsilon,D_x\varphi(t_\epsilon,x_\epsilon)\big).\qquad(2.10)$$

Then

$$\varphi_t(t_\epsilon,x_\epsilon)=\frac{t_\epsilon-s_\epsilon}{\epsilon}-\eta-\gamma\frac{1}{t_\epsilon^2},$$
$$D_x\varphi(t_\epsilon,x_\epsilon)=\frac{x_\epsilon-y_\epsilon}{\epsilon}+m\beta\langle x_\epsilon\rangle^{m-2}x_\epsilon,$$
$$\psi_s(s_\epsilon,y_\epsilon)=\frac{t_\epsilon-s_\epsilon}{\epsilon}+\eta+\gamma\frac{1}{s_\epsilon^2},$$
$$D_y\psi(s_\epsilon,y_\epsilon)=\frac{x_\epsilon-y_\epsilon}{\epsilon}-m\beta\langle y_\epsilon\rangle^{m-2}y_\epsilon,$$

and $(t_\epsilon,x_\epsilon),(s_\epsilon,y_\epsilon)\in K$and for $R>0$with $K\subset\bar B(0,R)$, we have

$$\|\,f(t_\epsilon,x_\epsilon,u)-f(s_\epsilon,y_\epsilon,u)\,\|\le L_f(R)(|\,t_\epsilon-s_\epsilon\,|+\|\,x_\epsilon-y_\epsilon\,\|).$$

In view of $\|\,D\xi(x)\,\|\le 1$, we have

$$\bar H\left(s_\epsilon,y_\epsilon,D_y\psi(s_\epsilon,y_\epsilon)\right)-\bar H\big(t_\epsilon,x_\epsilon,D_x\varphi(t_\epsilon,x_\epsilon)\big)\le$$
$$\le\sup_{u\in U}\left\{-f(s_\epsilon,y_\epsilon,u)\left(D_y\psi(s_\epsilon,y_\epsilon)+\alpha u_2(s_\epsilon,y_\epsilon)D\xi(y_\epsilon)\right)-e^{\alpha\xi(y_\epsilon)}\ell(s_\epsilon,y_\epsilon,u)\right\}-$$
$$-\sup_{u\in U}\big\{-f(t_\epsilon,x_\epsilon,u)\big(D_x\varphi(t_\epsilon,x_\epsilon)+\alpha u_1(t_\epsilon,x_\epsilon)D\xi(x_\epsilon)\big)-e^{\alpha\xi(x_\epsilon)}\ell(t_\epsilon,x_\epsilon,u)\big\}\le$$
$$\le L_f(R)(|\,t_\epsilon-s_\epsilon\,|+\|\,x_\epsilon-y_\epsilon\,\|)\left(\alpha\,\|\,u_1\,\|_\infty+\frac{\|\,x_\epsilon-y_\epsilon\,\|}{\epsilon}+\beta m\langle x_\epsilon\rangle^{m-1}\right)+$$
$$+C(1+\|\,y_\epsilon\,\|^p)[\beta m(\langle x_\epsilon\rangle^{m-1}+\langle y_\epsilon\rangle^{m-1})+\alpha\,|\,u_1(t_\epsilon,x_\epsilon)-u_2(s_\epsilon,y_\epsilon)\,|]+$$
$$+e^{-\alpha\xi(x_\epsilon)}\,|\,\ell(t_\epsilon,x_\epsilon,u)-\ell(s_\epsilon,y_\epsilon,u)\,|+C\,|\,e^{-\alpha\xi(y_\epsilon)}-e^{-\alpha\xi(x_\epsilon)}\,|\,(1+\|\,y_\epsilon\,\|^2).$$

Hence, from (2.10) it follows that

$$2\eta+\gamma\frac{1}{t_\epsilon^2}+\gamma\frac{1}{s_\epsilon^2}\le$$

$$\leq L_f(R)(\mid t_\epsilon - s_\epsilon \mid + \parallel x_\epsilon - y_\epsilon \parallel)\left(\alpha \parallel u_1 \parallel_\infty + \frac{\parallel x_\epsilon - y_\epsilon \parallel}{\epsilon} + \beta m\langle x_\epsilon\rangle^{m-1}\right) +$$
$$+C(1+\parallel y_\epsilon \parallel^p)[\beta m(\langle x_\epsilon\rangle^{m-1} + \langle y_\epsilon\rangle^{m-1}) + \alpha \mid u_1(t_\epsilon, x_\epsilon) - u_2(s_\epsilon, y_\epsilon) \mid] +$$
$$+e^{-\alpha\xi(x_\epsilon)} \mid \ell(t_\epsilon, x_\epsilon, u) - \ell(s_\epsilon, y_\epsilon, u) \mid + C \mid e^{-\alpha\xi(y_\epsilon)} - e^{-\alpha\xi(x_\epsilon)} \mid (1+\parallel y_\epsilon \parallel^2). \quad (2.11)$$

In view of $\beta(\langle x_\epsilon\rangle^m + \langle y_\epsilon\rangle^m) \leq L\ (m, \leq m_0\beta \leq \beta_0)$, it follows from (2.11) that

$$2\eta + \gamma\frac{1}{t_\epsilon^2} + \gamma\frac{1}{s_\epsilon^2} \leq L_f(R)(\mid t_\epsilon - s_\epsilon \mid + \parallel x_\epsilon - y_\epsilon \parallel)\left(\alpha \parallel u_1 \parallel_\infty + \frac{\parallel x_\epsilon - y_\epsilon \parallel}{\epsilon} + \beta m\langle x_\epsilon\rangle^{m-1}\right) +$$
$$1+\parallel y_\epsilon \parallel^p \leq 1 + (L/\beta)^{\frac{p}{m_0}} = \bar{C}.$$

Now, by taking $m \leq \min\left\{\frac{\eta}{2CCL}, m_0\right\}$and $\alpha \leq \frac{\eta}{2CCL}$, it follows from (2.12) that

$$2\eta + \gamma\frac{1}{t_\epsilon^2} + \gamma\frac{1}{s_\epsilon^2} \leq \eta + O(\epsilon), \quad (2.12)$$

which is a contradiction. Thus, it follows that

$$u_1(t,x) \leq u_2(t,x), (t,x) \in [0,T] \times R^n.$$

By using

$$u_1(t,x) = v_1(t,x)e^{-\alpha\xi(x)} \leq u_2(t,x) = v_2(t,x)e^{-\alpha\xi(x)},$$

we have

$$v_1(t,x) \leq v_2(t,x), (t,x) \in [0,T] \times R^n.$$

From Theorem 2.3, it follows that the value function $v(t,x)$ is the unique, unbounded and continuous viscosity solution of HJB equation terminal problem (2.5) and (2.6).

3. APPROXIMATION OF THE VISCOSITY SOLUTION OF THE HAMILTON-JACOBI-BELLMAN EQUATION

3.1. Time Approximation Schemes

For any $t \in [0,T)$ and any natural number $N \geq [T] + 1$ we set $h = \frac{T-t}{N}(0 < h < 1)$, where $[a]$denotes the largest integer that is not greater than a.

We discretize the optimal control problem (2.1) using Euler implicit scheme with respect to time variable.

$$\begin{cases} y_{j+1} = y_j + hf\big(t_j, (y_j + y_{j+1}),/2u_j\big), & j = 0, \dots, N-1 \\ y_0 = x, & x \in R^n, \end{cases} \quad (3.1)$$

$$J_h\big(t, x, u_j\big) = \sum_{j=0}^{N-1} h\,\ell\big(t_j, y_j, u_j\big) + g(y_N) \to \min_{\{u_j\}}, \quad (3.2)$$

where $y_j = y\big(t_j\big), u_j = u\big(t_j\big)$. Then, the value function of (3.1) and (3.2) is defined as

$$v_h(t,x) = \inf_{\{u_j\}} J_h\big(t, x, \{u_j\}\big).$$

Theorem 3.1. Suppose that the assumptions (A1)-(A4) hold. Then for $h(0 < h < h_0)$and the value function $v_h(t,x)$, we have

$$\exists C_1 > 0, \forall t \in [0,T], \forall x \in R^n, \mid v_h(t,x) \mid \leq C_1(1+\parallel x \parallel^2). \quad (3.3)$$

Proof. For control sequence $\{u_j\}$, we have

$$\mid v_h(t,x) \mid \leq \sum_{j=0}^{N-1} h \mid \ell\big(t_j, y_j, u_j\big) \mid + \mid g(y_N) \mid \leq \sum_{j=0}^{N-1} h\,C\big(1+\parallel y_j \parallel^2\big) + C(1+\parallel y_N \parallel^2). \quad (3.4)$$

From the assumption (A2) it follows that

$$\left(y_{j+1}, \frac{y_j + y_{j+1}}{2}\right) = \left(y_j, \frac{y_j + y_{j+1}}{2}\right) + h\left(f, \left(t_j, \frac{y_j + y_{j+1}}{2}, u_j\right)\frac{y_j + y_{j+1}}{2}\right),$$

$$\| y_{j+1} \|^2 \leq \| y_j \|^2 + 2Ch\left(1 + \| y_j + y_{j+1} \|^2/4\right),$$
$$\| y_{j+1} \|^2 \leq \| y_j \|^2 + 2Ch + Ch\left(\| y_j \|^2 + \| y_{j+1} \|^2\right),$$

which implies that

$$\| y_{j+1} \|^2 \leq \frac{2Ch}{1-Ch} + \frac{1+Ch}{1-Ch} \| y_j \|^2 = \| y_j \|^2 + \frac{2Ch}{1-Ch}\left(1 + \| y_j \|^2\right). \tag{3.5}$$

Now, if we choose $h_0 = \min\left(1, \frac{1}{2C}\right)$, for $h(0 < h < h_0)$it follows from (3.5) that

$$\| y_{j+1} \|^2 \leq \| y_j \|^2 + 4Ch\left(1 + \| y_j \|^2\right), j = 0, \dots, N-1$$

and then

$$\| y_1 \|^2 \leq \| x \|^2 + 4Ch(1 + \| x \|^2) \leq (1 + 4Ch)(1 + \| x \|^2),$$
$$\| y_2 \|^2 \leq \| y_1 \|^2 + 4Ch(1 + \| y_1 \|^2) \leq$$
$$\leq \| x \|^2 + 4Ch(1 + \| x \|^2) + 4Ch(1 + \| x \|^2) + 4Ch(1 + \| x \|^2) \leq$$
$$\leq (1 + 4Ch)(1 + \| x \|^2) + 4Ch(1 + \| x \|^2) + 4Ch(1 + \| x \|^2) \leq$$
$$\leq (1 + 4Ch)^2(1 + \| x \|^2),$$

which gives us the following inequality

$$\| y_j \|^2 \leq (1 + 4Ch)^j(1 + \| x \|^2) \leq e^{4Cjh}(1 + \| x \|^2) \leq e^{4CT}(1 + \| x \|^2), j = 1, \bar{N}. \tag{3.6}$$

Thus, we have

$$| v_h(t,x) | \leq \sum_{j=0}^{N-1} h\, C\left(1 + e^{4CT}(1 + \| x \|^2)\right) + C_0\left(1 + e^{4CT}(1 + \| x \|^2)\right) \leq$$
$$\leq CT\left(1 + e^{4CT}(1 + \| x \|^2)\right) + C_0\left(1 + e^{4CT}(1 + \| x \|^2)\right) \leq \bar{C}_1(1 + \| x \|^2),$$

where $\bar{C}_1 = (C_0 + CT)(1 + e^{4CT})$. Then, the equation (3.3) holds. □

Remark 3.2. For the control system (2.1), if the time variable is discretized by Euler scheme, there is difficulty in the proof of Theorem 3.1. Therefore, the time discretization is based on Euler's implicit scheme for the control system (2.1).

In (3.1), for $h(0 < h < h_0)$if we use Euler scheme

$$y_{j+1} \approx y_j + hf\left(t_j, y_j, u_j\right),$$

the scheme (3.1) can be rewritten as follows.

$$\begin{cases} y_{j+1} = y_j + hf\left(t_j, y_j, + \frac{h}{2} f(t_j, y_j, u_j) u_j\right), & j = 0, \dots, N-1, \\ y_0 = x, & x \in R^n. \end{cases} \tag{3.7}$$

We will use the time discretization scheme (3.7) in proving the convergence of the approximation scheme when the discretization step goes to zero.

Lemma 3.3. *The value function $v_h$satisfies the discretized HJB equation*

$$v_h(t,x) = \inf_{u \in U} \left\{ h\ell(t,x,u) + v_h\left(t, +hx + hf\left(t, x, + \frac{h}{2} f(t,x,u) u\right)\right)\right\}$$
$$v_h(T,x) = g(x).$$

Lemma 3.4. *Suppose that the assumptions (A1)-(A4) hold. Then for $h \in (0, h_0)$and for any compact subset $K \subset R^n$, we have*

$$\exists \bar{C}_K > 0, \forall t, \bar{t} \in [0,T], \forall x, z \in K,$$
$$| v_h(t,x) - v_h(\bar{t}, z) | \leq \bar{C}_K(| t - \bar{t} | + \| x - z \|). \tag{3.8}$$

Proof. Fix any $t \in [0,T]$and let $x, z \in K \subset \bar{B}(0,R)$. It follows from the definition of the value function that

$$\forall \epsilon > 0, \exists \{\bar{u}_j\}, v_h(t,z) \geq J_h\left(t, z, \{\bar{u}_j\}\right) - \epsilon.$$

Now, let $\{y_j\}$and $\{z_j\}$denote trajectories corresponding to $\{\bar{u}_j\}$which start at $x$and $z \in K$, respectively. Then, from (3.6) we have

$$\| y_j \| \leq e^{2CT}(1+\| x \|) \text{ and } \| z_j \| \leq e^{2CT}(1+\| z \|), j = 1,\bar{N}.$$

Thus, by taking $\bar{R} = e^{2CT}(1+R)$, we have

$$y_j \in B(0,\bar{R}), j = 0,\bar{N}, z_j \in B(0,\bar{R}), j = 0,\bar{N}.$$

For trajectories $\{y_j\}$and $\{z_j\}$we have

$$y_1 = x + hf\left(t, x, +\frac{h}{2}f(t,x,\bar{u}_0)\bar{u}_0\right),$$

$$z_1 = z + hf\left(t, z, +\frac{h}{2}f(t,z,\bar{u}_0)\bar{u}_0\right)$$

and from the assumption (A1) it follows that

$$\| y_1 - z_1 \| \leq \| x - z \| + L_f(\bar{R})\left(\| x - z \| + \frac{1}{2}hL_f(\bar{R}) \| x - z \|\right).$$

Thus, if $h \leq \bar{h} = \min\left(h_0, \frac{1}{L_f(\bar{R})}\right)$, it follows that

$$\| y_1 - z_1 \| \leq \left(1 + 2hL_f(\bar{R})\right) \| x - z \|.$$

In the same manner, we obtain

$$\| y_2 - z_2 \| \leq \left(1 + 2hL_f(\bar{R})\right) \| y_1 - z_1 \| \leq \left(1 + 2hL_f(\bar{R})\right)^2 \| x - z \|$$

and in general, we obtain

$$\| y_j - z_j \| \leq \left(1 + 2hL_f(\bar{R})\right)^j \| x - z \| \leq e^{2jhL_f(\bar{R})} \| x - z \|, j = 1, \dots, N.$$

Therefore, we have

$$v_h(t,x) - v_h(t,z) \leq \sum_{j=0}^{N-1} h\,\ell\left(t_j, y_j, \bar{u}_j\right) - \sum_{j=0}^{N-1} h\,\ell\left(t_j, z_j, \bar{u}_j\right) + g(y_N) - g(z_N) + \epsilon \leq$$

$$\leq \sum_{j=0}^{N-1} h\,L_\ell(\bar{R}) \| y_j - z_j \| + L_g(\bar{R}) \| y_N - z_N \| + \epsilon \leq$$

$$\leq \sum_{j=0}^{N-1} h\,L_\ell(\bar{R}) e^{2jhL_f(\bar{R})} \| x - z \| + L_g(\bar{R}) e^{2NhL_f(\bar{R})} \| x - z \| + \epsilon \leq$$

$$\leq L_\ell(\bar{R}) T e^{2L_f(\bar{R})T} \| x - z \| + L_g(\bar{R}) e^{2L_f(\bar{R})T} \| x - z \| + \epsilon \leq$$

$$\leq \left(L_\ell(\bar{R})T + L_g(\bar{R})\right) e^{2L_f(\bar{R})T} \| x - z \| + \epsilon \leq \bar{C}_R \| x - z \| + \epsilon,$$

where $\bar{C}_R = \left(L_\ell(\bar{R})T + L_g(\bar{R})\right) e^{2L_f(\bar{R})T}$. Even if role of $x$and $z$is exchanged, the above inequality remains true and then we have

$$| v(t,x) - v(t,z) | \leq \bar{C}_R \| x - z \|. \tag{3.9}$$

Now, for any $\bar{t} \in [0,T]$suppose that $\bar{t} \leq t$. Put

$$j_t = \left\lceil \frac{t - \bar{t}}{h} \right\rceil + 1.$$

Then, for control sequence $\{u_j\}$it follows from Lemma 3.3 and (3.9) that

$$v_h(\bar{t},x) - v_h(t,z) \leq \sum_{j=0}^{j_t-1} h\,\ell\left(t_j, y_j, u_j\right) + v_h\left(t, y_{j_t}\right) - v_h(t,z) \leq$$

$$\leq \sum_{j=0}^{j_t-1} h\,| \ell\left(t_j, y_j, u_j\right) | + | v_h\left(t, y_{j_t}\right) - v_h(t,x) | + | v_h(t,x) - v_h(t,z) | \leq$$

$$\leq \sum_{j=0}^{j_t-1} h \mid \ell(t_j, y_j, u_j) \mid + \bar{C}_R \parallel y_{j_t} - x \parallel + \bar{C}_R \parallel x - z \parallel,$$

where $y_{j_t} = y_{j_t-1} + (t - t_{j_t-1}) f\left(t_{j_t-1}, (y_{j_t-1} + y_{j_t}), /2u_{j_t-1}\right)$. Thus, for any $x, z \in K$ and any $\bar{t} \in [0,T]$with $\bar{t} \leq t$, if we take

$$M_{\bar{R}f} = \max_{[0,T]\times\bar{B}(0,\bar{R})\times U} f(t,x,u), M_{\bar{R}\ell} = \max_{[0,T]\times\bar{B}(0,\bar{R})\times U} \ell(t,x,u),$$

we have

$$v_h(\bar{t}, x) - v_h(t, z) \leq \sum_{j=0}^{j_t-1} h\, M_{\bar{R}\ell} + \bar{C}_R \sum_{j=0}^{j_t-1} h\, M_{\bar{R}f} + \bar{C}_R \parallel x - z \parallel \leq$$

$$\leq \left(M_{\bar{R}\ell} + \bar{C}_R M_{\bar{R}f}\right) \mid t - \bar{t} \mid + \bar{C}_R \parallel x - z \parallel \leq \bar{C}_K(\parallel x - z \parallel + \mid t - \bar{t} \mid),$$

where $\bar{C}_K = \max\{M_{\bar{R}\ell}, + \bar{C}_R M_{\bar{R}f} \bar{C}_R\}$. Also, for any $\bar{t} \in [0,T]$with $t \leq \bar{t}$, the proof is similar. If the role of $x$and $z$is exchanged, the above inequality remains true and then we have (3.8).

Theorem 3.5. *Suppose that the assumptions (A1)-(A4) hold. Let* $v(t,x)$*be the value function of the optimal control problem (2.1). Then, for any compact subset* $K \subset R^n$*, the value function* v_h*converges uniformly to* v*in* $[0,T] \times K$*as* $h \to 0$.

Proof. It follows from Theorem 3.1 and Lemma 3.4 that $\{v_h\}$is uniformly bounded in $[0,T] \times K$and Equi continuous for any compact set $K \subset R^n (D \subset K)$. Hence, by Ascoli-Arzela's theorem, there exists a uniformly convergent subsequence of $\{v_h\}$at $[0,T] \times K$as $h \to 0$. We denote the subsequence by $v_h$again without loss of generality. Then, $\lim_{h\to 0} v_h = \bar{v}$.

Now, let us show that $\bar{v}(t,x)$is a viscosity solution of (2.5). First, we prove that $\bar{v}(t,x)$is a viscosity sub-solution of (2.5) at $(0,T) \times R^n$. For any $\psi \in C^1\left((0,T) \times R^n\right)$, let $(t_0, x_0) \in (0,T) \times R^n$ be a strict local maximum point of $\bar{v} - \psi$. For some $r > 0$, we can choose a sequence $\{(t_h, x_h)\} \in B_r(t_0, x_0)$of maximums of $v_h - \psi$in $B_r(\bar{t_0}, x_0)$such that

$$\lim_{h\to 0}(t_h, x_h) = (t_0, x_0), \lim_{h\to 0} v_h(t_h, x_h) = \bar{v}(t_0, x_0),$$

where $B_r(t_0, x_0)$is open sphere with radius $r > 0$centered at point (t_0, x_0)[1].

Now, by taking $h$sufficiently small, we can show that

$$\forall u \in U, \left(t_h, + h x_h + h f\left(t_h, x_h, + \frac{h}{2} f(t_h, x_h, u) u\right)\right) \in B_r(\bar{t_0}, x_0).$$

Thus, it holds that

$$v_h(t_h, x_h) - \psi(t_h, x_h) \geq$$

$$\geq v_h\left(t_h, + h x_h + h f\left(t_h, x_h, + \frac{h}{2} f(t_h, x_h, u) u\right)\right) - \psi\left(t_h, + h x_h + h f\left(t_h, x_h, + \frac{h}{2} f(t_h, x_h, u) u\right)\right)$$

By Lemma 3.3, since

$$v_h(t_h, x_h) \leq h\ell(t_h, x_h, u) + v_h\left(t_h, + h x_h + h f\left(t_h, x_h, + \frac{h}{2} f(t_h, x_h, u) u\right)\right),$$

we have

$$0 \geq v_h(t_h, x_h) - h\ell(t_h, x_h, u) - v_h\left(t_h, + h x_h + h f\left(t_h, x_h, + \frac{h}{2} f(t_h, x_h, u) u\right)\right) \geq$$

$$\geq \psi(t_h, x_h) - \psi\left(t_h, + h x_h + h f\left(t_h, x_h, + \frac{h}{2} f(t_h, x_h, u) u\right)\right) - h\ell(t_h, x_h, u). \qquad (3.10)$$

By dividing the both sides by $h$and by taking $h \to 0$, we have

$$-\frac{\partial \psi}{\partial t}(t_0, x_0) - f(t_0, x_0, u) \cdot D_x \psi(t_0, x_0) - \ell(t_0, x_0, u) \leq 0.$$

The arbitrariness of $u$implies that

$$-\frac{\partial\psi}{\partial t}(t_0,x_0)+\sup_{u\in U}\{-f(t_0,x_0,u)\cdot D_x\psi(t_0,x_0)-\ell(t_0,x_0,u)\}\geq 0.$$

Thus, $\bar{v}(t,x)$is a viscosity sub-solution of (2.5) in $(0,T)\times R^n$.

Similarly, we can prove that $\underline{v}(t,x)$is a viscosity super-solution of (2.5).

3.2. Time-Space Approximation Schemes

Let us divide any bounded convex set $\Omega\subset R^n$into simplices as follows. Let $T_j(j,=,1\ldots m)$denote set of simplices of R^nsuch that $\Omega=\bigcup_{j=1}^{m}T_j$, $\mathring{T}_j\cap\mathring{T}_i=\emptyset,(j\neq i)$and $x_i(i=1,\bar{M})$be nodes of T_j. Let

$$k=\max_j\{diamT_j\},$$
$$S=\{t_v,|,v,=01\ldots N\},\ G=\{x_j,|,j,=12\ldots M\},$$

where $diamT_j$denotes the diameter of the set T_j. Consider the following problem:

$$\begin{cases}v_h^k(t_v,x_i)=\inf_{u\in U}\left\{h\ell(t_v,x_i,u)+v_h^k\left(t_{v+1},x_i+hf\left(t_v,x_i,+\frac{h}{2}f(t_v,x_i,u)u\right)\right)\right\}\\ v_h^k(T,x_i)=g(x_i),\ i=1,\bar{M}.\end{cases}\quad(3.11)$$

Fix any $u\in U$and suppose that $\bar{x}\in\bar{\Omega}$. The linear convex combination

$$x_i+hf\left(t_v,x_i,+\frac{h}{2}f(t_v,x_i,u)u\right)=\sum_{j=1}^{M}\lambda_i^j\,x_j$$

$$\sum_{j=1}^{M}\lambda_i^j=1,0\leq\lambda_i^j\leq 1$$

is unique in each simplex. Now, for $(t_v,x_i)\in S\times G$, define a continuous piecewise linear function $v_h^k\in C([0,T]\times\bar{\Omega})$as follows.

$$\begin{cases}x_i+hf\left(t_v,x_i,+\frac{h}{2}f(t_v,x_i,u)u\right)=\sum_{j=1}^{M}\lambda_i^j\,x_j, & i=1,\bar{M},v=0,\bar{N-1}\\ \sum_{j=1}^{M}\lambda_i^j=1,0\leq\lambda_i^j\leq 1\\ v_h^k(t_v,x_i)=\min_{u\in U}\left\{h\ell(t_v,x_i,u)+\sum_{j=1}^{M}\lambda_i^j\,v_h^k(t_{v+1},x_j)\right\}\\ v_h^k(T,x_i)=g(x_i), & i=1,\bar{M}.\end{cases}\quad(3.12)$$

We define this function v_h^kas the solution of (3.11).

Lemma 3.6. *Suppose that the assumptions (A1)-(A4) hold. Let $v_h$be a value function of time-discretized problem and $v_h^k(t,x)$be a solution of (3.11). Then for any $(t_j,x)\in S\times\bar{\Omega}$we have*

$$\exists\bar{C}_2>0,|\,v_h^k(t_j,x)-v_h(t_j,x)\,|\leq\bar{C}_2\left(\frac{k}{h}+k\right),j=0,\bar{N}.\quad(3.13)$$

Proof. For any $(t_j,x)\in S\times\bar{\Omega}$, suppose that x lies in some simplex T_x. If we denote $I_x=\{i,|,x_i,\in T_x i=12\ldots M\}$, we have

$$x=\sum_{i\in I_x}\lambda_i\,x_i,\|\,x-x_i\,\|\leq k,i\in I_x,\sum_{i\in I_x}\lambda_i=1.$$

Then, it holds that

$$v_h^k\left(t_j,\textstyle\sum_{i\in I_x}\lambda_i\,x_i\right)=\sum_{i\in I_x}\lambda_i\,v_h^k(t_j,x_i),$$

so we have

$$|\,v_h^k(t_j,x)-v_h(t_j,x)\,|\leq\sum_{i\in I_x}\lambda_i\,|\,v_h^k(t_j,x_i)-v_h(t_j,x_i)\,|+\sum_{i\in I_x}\lambda_i\,|\,v_h(t_j,x_i)-v_h(t_j,x)\,|.\quad(3.14)$$

Then, it follows from Lemma 3.4 that

$$\exists \bar{C}_R > 0, \mid v_h(t_j, x_i) - v_h(t_j, x) \mid \leq \bar{C}_R \parallel x - x_i \parallel \leq \bar{C}_R k, j = \overline{0, N}. \tag{3.15}$$

Because it holds

$$\mid v_h^k(T, x) - v_h(T, x) \mid \leq \sum_{i \in I_x} \lambda_i \mid g(x_i) - g(x) \mid \leq L_g(R) \parallel x_i - x \parallel = L_g(R)k,$$

it follows from (3.14) and (3.15) that

$$\mid v_h^k(t_j, x) - v_h(t_j, x) \mid \leq \sum_{i \in I_x} \lambda_i \mid v_h^k(t_j, x_i) - v_h(t_j, x_i) \mid + \sum_{i \in I_x} \lambda_i \mid v_h(t_j, x_i) - v_h(t_j, x) \mid$$

$$\leq \sum_{i \in I_x} \lambda_i \mid v_h^k(t_j, x_i) - v_h(t_j, x_i) \mid + \bar{C}_R k =$$

$$= \bar{C}_R k + \sum_{i \in I_x} \lambda_i \left\{ \left| \min_u \left[h\ell(t_j, x_i, u) + v_h^k \left(t_{j+1}, x_i + hf\left(t_j, x_i, + \frac{h}{2} f(t_j, x_i, u)u\right)\right)\right] - \right.\right.$$

$$\left.\left. - \min_u \left[h\ell(t_j, x_i, u) + v_h \left(t_{j+1}, x_i + hf\left(t_j, x_i, + \frac{h}{2} f(t_j, x_i, u)u\right)\right)\right]\right| \right\} \leq$$

$$\leq \bar{C}_R k + \max_u \mid v_h^k \left(t_{j+1}, x_i + hf\left(t_j, (x + \bar{x}),/2u\right)\right) - v_h \left(t_{j+1}, x_i + hf\left(t_j, (x_i, \bar{x}),/2u\right)\right) \mid \leq$$

$$\leq \bar{C}_R k + \max_{\bar{x}_{i+1} \in \bar{\Omega}} \mid v_h^k(t_{j+1}, \bar{x}_{i+1}) - v_h(t_{j+1}, \bar{x}_{i+1}) \mid \leq$$

$$\leq 2\bar{C}_R k + \max_{\bar{x}_{i+2} \in \bar{\Omega}} \mid v_h^k(t_{j+2}, \bar{x}_{i+2}) - v_h(t_{j+2}, \bar{x}_{i+2}) \mid \leq$$

$$\leq (N - j)\bar{C}_R k + \max_{\bar{x}_{i+N-j} \in \bar{\Omega}} \mid v_h^k(T, \bar{x}_{i+N-j}) - v_h(T, \bar{x}_{i+N-j}) \mid \leq$$

$$\leq (N - j)\bar{C}_R k + L_g(R)k \leq T\bar{C}_R \frac{k}{h} + L_g(R)k \leq \bar{C}_2 \left(\frac{k}{h} + k \right),$$

where $\bar{C}_2 = \max\{T, \bar{C}_R L_g(R)\}$.

Theorem 3.7. *Suppose that the assumptions (A1)-(A4) hold. Let $v(t, x)$be the value function of optimal control problem (2.1) and let $K = O(h^{1+\gamma})$, $0 < \gamma < 1$. Then*

$$\lim_{h \to 0} v_h^k = v \text{ uniformly on } [0, T] \times \Omega.$$

Proof. For any $(t, x) \in [0, T] \times \bar{\Omega}$, we obtain $t \in [t_{v-1}, t_v]$, $v \in \overline{1, N}$. Then it follows from Theorem 3.5 and Lemma 3.6 that

$$\mid v_h^k(t_v, x) - v(t, x) \mid \leq \mid v_h^k(t_v, x) - v_h(t_v, x) \mid + \mid v_h(t_v, x) - v_h(t, x) \mid + \mid v_h(t_v, x) - v(t, x) \mid \leq$$

$$\leq \bar{C}_2 \left(\frac{k}{h} + k \right) + \bar{C}_R h + \mid v_h(t, x) - v(t, x) \mid. \tag{3.16}$$

Now, by taking $k = O(h^{1+\gamma})$, from (3.16) and Theorem 3.5 we have

$$\mid v_h^k(t, x) - v(t, x) \mid \to 0, (h \to 0).$$

It follows from Theorem 3.7 that even if the boundedness condition for functions involved in control system and cost functional is replaced with the growth condition, the solution $v_h^k$of time-space discretized HJB equation (3.11) converges uniformly to the viscosity solution of the HJB equation (2.5) in $[0, T] \times \bar{\Omega}$as $h \to 0$.

4. CONVERGENCE OF APPROXIMATE SCHEME OF OPTIMAL FEEDBACK CONTROL

Now, for any $(t, x) \in [0, T) \times \bar{\Omega}$we define $u_h^k(t, x)$ as

$$u_h^k(t, x) = \arg\min_{u \in U} \left\{ h\ell(t, x, u) + v_h^k \left(t, + hx + hf\left(t, x, + \frac{h}{2} f(t, x, u)u\right)\right)\right\}. \tag{4.1}$$

Then, let sequences $\{u_{h,j}^k\} = \{u_h^k(t_j, y_j)\}$and $\{y_j\}$satisfy the following equations.

$$\begin{cases} y_{j+1} = y_j + hf\left(t_j, y_j, +\frac{h}{2} f\left(t_j, y_j, u_h^k(t_j, y_j)\right) u_h^k(t_j, y_j)\right), & j = 0, \dots, N-1 \\ y_0 = x, & x \in R^n. \end{cases} \tag{4.2}$$

Define

$$J_h^k(t, x, \{u_{h,j}^k\}) = \sum_{j=0}^{N-1} h\, \ell(t_j, y_j, u_{h,j}^k) + g(y_N). \tag{4.3}$$

and define the feedback control by

$$u_h^k(\tau) = u_{h,j}^k, \tau \in [t_j, t_{j+1}), j = 0, \bar{N} - 1 \tag{4.4}$$

Lemma 4.1. Let $\{u_h^k\}$be defined by (4.1) and (4.2), and $u_h^k(\cdot)$be the feedback control defined by (4.4). Then for any $(t, x) \in [0, T] \times \bar{\Omega}$, we have

$$\exists \bar{C}_3 > 0, \mid J\left(t, x, u_h^k(\cdot)\right) - J_h^k(t, x, \{u_h^k\}) \mid \leq \bar{C}_3 h. \tag{4.5}$$

Proof. Fix any $t \in [0, T]$ and let $x \in \bar{\Omega} \subset \bar{B}(0, R)$. It follows from (3.6) that the trajectory $\{y_j\}$corresponding to the control sequence $\{u_h^k\}$defined by (4.1) and (4.2) satisfies

$$\| y_j \| \leq e^{2CT}(1 + \| x \|), j = 1, \bar{N}.$$

Thus, for $\bar{R} = e^{2CT}(1 + R)$, we obtain

$$y_j \in B(0, \bar{R}), j = 0, \bar{N}.$$

For any $(t, x) \in [0, T] \times \bar{\Omega}$we have

$$\mid J\left(t, x, u_h^k(\cdot)\right) - J_h^k(t, x, \{u_h^k\}) \mid \leq$$

$$\leq \left| \int_t^T \ell\left(\tau, y, (\tau) u_h^k(\tau)\right) d\tau + g\left(y(T)\right) - \sum_{j=0}^{N-1} h\, \ell(t_j, y_j, u_{h,j}^k) - g(y_N) \right| \leq$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \mid \ell\left(\tau, y, (\tau) u_h^k(\tau)\right) - \ell(t_j, y_j, u_{h,j}^k) \mid d\tau + \mid g\left(y(T)\right) - g(y_N) \mid \leq$$

$$\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} L_\ell(\bar{R})\left(\mid \tau - t_j \mid + \| y(\tau) - y_j \|\right) d\tau + L_g(\bar{R}) \| y(T) - y_N \| \leq$$

$$\leq L_\ell(\bar{R})(T - t)h + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} L_\ell(\bar{R})\left(\mid \tau - t_j \mid + \| y(\tau) - y_j \|\right) d\tau + L_g(\bar{R}) \| y(T) - y_N \| \tag{4.6}$$

and

$$y(s) = y(t_j) + \int_{t_j}^{s} f\left(\tau, y, (\tau) u_h^k(\tau)\right) d\tau, s \in [t_j, t_{j+1}),$$

$$y_{j+1} = y_j + hf\left(t_j, y_j, +\frac{h}{2} f(t_j, y_j, u_{h,j}^k) u_{h,j}^k\right), j = 0, \bar{N} - 1.$$

Now, let $z(s) = y_j + \frac{h}{2} f(t_j, y_j, u_{h,j}^k), s \in [t_j, t_{j+1}), j = 0, \bar{N} - 1$. Then, we have

$$\| y(t_j) - y_j \| \leq \| y(t_{j-1}) - y_{j-1} \| +$$

$$+ \int_{t_{j-1}}^{t_j} \| f(\tau, y, (\tau) u_{h,j-1}^k) - f\left(t_{j-1}, y_{j-1}, +\frac{h}{2} f(t_{j-1}, y_{j-1}, u_{h,j-1}^k) u_{h,j-1}^k\right) \| d\tau \leq$$

$$\leq \| y(t_{j-1}) - y_{j-1} \| + L_f(\bar{R}) \int_{t_{j-1}}^{t_j} (\mid \tau - t_{j-1} \mid + \| y(\tau) - z(\tau) \|) d\tau, j = 1, \bar{N}.$$

Then, for $s \in [t_j, t_{j+1}), j = 0, \bar{N-1}$, we have

$$\| y(s) - z(s) \| = \left\| y(s) - y_j - \frac{h}{2} f(t_j, y_j, u_{h,j}^k) \right\| \leq \| y(s) - y_{j+1} \| + M_f h \leq$$

$$\leq \| y(t_j) - y_j \| + \int_{t_j}^{s} \| f(\tau, y, (\tau) u_{h,j}^k) - f(t_j, z, (\tau) u_{h,j}^k) \| d\tau + 2M_f h \leq$$

$$\leq \| y(t_j) - y_j \| + L_f(\bar{R}) \int_{t_j}^{s} (| \tau - t_j | + \| y(\tau) - z(\tau) \|) d\tau + 2M_f h \leq$$

$$\leq \| y(t_j) - y_j \| + L_f(\bar{R}) h(s - t_j) + L_f(\bar{R}) \int_{t_j}^{s} \| y(\tau) - z(\tau) \| d\tau + 2M_f h \leq$$

$$\leq \| y(t_{j-1}) - y_{j-1} \| + L_f(\bar{R}) h(s - t_{j-1}) + L_f(\bar{R}) \int_{t_{j-1}}^{s} \| y(\tau) - z(\tau) \| d\tau + 2M_f h \leq$$

$$\leq \cdots \leq$$

$$L_f(\bar{R}) h(s - t) + L_f(\bar{R}) \int_{t}^{s} \| y(\tau) - z(\tau) \| d\tau + 2M_f h,$$

where $M_f = \max_{(t,x,u) \in [0,T] \times \bar{\Omega} \times U} \| f(t, x, u) \|$. Thus, from Gronwall-inequality, we obtain

$$\| y(s) - z(s) \| \leq \left(L_f(\bar{R}) T + 2M_f \right) h e^{T L_f(\bar{R})} = C_R h, s \in [t_j, t_{j+1}], j = 0, \bar{N-1}, \tag{4.7}$$

$$\| y(s) - z(s) \| \leq \| y(s) - y_{N-1} \| + \| y_{N-1} - y_N \| \leq C_R h + M_f h = (C_R + M_f) h,$$

where $C_R = \left(L_f(\bar{R}) T + 2M_f \right) e^{T L_f(\bar{R})}$. Also, for $s \in [t_{N-1}, t_N]$we have

$$\| y(s) - y_N \| \leq \| y(s) - y_{N-1} \| + \| y_{N-1} - y_N \| \leq C_R h + M_f h = (C_R + M_f) h,$$

and it follows from (4.6) that

(4.6)

$$| J\left(t, x, u_h^k(\cdot)\right) - J_h^k\left(t, x, \{u_h^k\}\right) | \leq$$

$$L_\ell(\bar{R})(T - t)h + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} L_\ell(\bar{R}) \| y(\tau) - y_j \| d\tau + L_g(\bar{R}) \| y(T) - y_N \| \leq$$

$$\leq \left(T L_\ell(\bar{R}) + T L_\ell(\bar{R}) C_R + L_g(\bar{R}) \left(C_R + M_f \right) \right) h = \bar{C}_3 h,$$

where $\bar{C}_3 = T L_\ell + T L_\ell C_T + L_g(\bar{R}) \left(C_R + M_f \right)$.

Lemma 4.2. *Let* $\{u_h^k\}$*be the control sequence defined by (4.1) and (4.2). Then, for any* $(t, x) \in [0, T] \times \bar{\Omega}$*, we have*

$$\exists \bar{C}_4 > 0, | J_h^k\left(t, x, \{u_h^k\}\right) - v_h(t, x) | \leq \bar{C}_4 \frac{k}{h^2}. \tag{4.8}$$

Proof. For any $(t, x) \in [0, T] \times \bar{\Omega}$we obtain

$$v_h(t, x) \leq J_h\left(t, x, \{u_{h,j}^k\}\right) = \sum_{j=0}^{N-1} h\, \ell\left(t_j, y_j, u_{h,j}^k\right) + g(y_N).$$

Now, we introduce the following notations

$$p_h(t_j, y_j, u) = \left\{ h\ell(t_j, y_j, u) + v_h\left(t_{j+1}, y_j + hf\left(t_j, y_j, + \frac{h}{2} f(t_j, y_j, u) u \right) \right) \right\},$$

$$p_h^k(t_j, y_j, u) = \left\{ h\ell(t_j, y_j, u) + v_h^k\left(t_{j+1}, y_j + hf\left(t_j, y_j, + \frac{h}{2} f(t_j, y_j, u) u \right) \right) \right\}.$$

From

$$p_h^k\left(t_j, y_j, u_{h,j}^k\right) = \min_{u \in U} p_h^k(t_j, y_j, u), v_h(t_N, y_N) = g(y_N),$$

it follows that

$$0 \le J_h^k(t,x,\{u_{h,j}^k\}) - v_h(t,x) = \sum_{j=0}^{N-1} h\,\ell(t_j,y_j,u_{h,j}^k) + g(y_N) - v_h(t,x) =$$

$$= \sum_{j=0}^{N-1} [\,p_h^k(t_j,y_j,u_{h,j}^k) - v_h^k(t_{j+1},y_{j+1})] + g(y_N) -$$

$$- \sum_{j=0}^{N-1} [\,v_h(t_j,y_j) - v_h(t_{j+1},y_{j+1})] - v_h(t_N,y_N) =$$

$$= \sum_{j=0}^{N-1} [\,v_h(t_{j+1},y_{j+1}) - v_h^k(t_{j+1},y_{j+1}) + \left(p_h^k(t_j,y_j,u_{h,j}^k) - v_h(t_j,y_j)\right)\Big].$$

Now, if we assume that

$$v_h(t_j,y_j) = p_h(t_j,y_j,u_{h,j}) = \min_{u\in U} p_h(t_j,y_j,u),$$

we have

$$0 \le J_h(t,x,\{u_{h,j}^k\}) - v_h(t,x) =$$

$$= \sum_{j=0}^{N-1} [\,v_h(t_{j+1},y_{j+1}) - v_h^k(t_{j+1},y_{j+1}) + \left(p_h^k(t_j,y_j,u_{h,j}^k) - p_h(t_j,y_j,u_{h,j})\right)\Big].$$

Then, it follows that

$$p_h^k(t_j,y_j,u_{h,j}^k) - p_h^k(t_j,y_j,u_{h,j}) \le 0$$

and

$$p_h(t_j,y_j,u_{h,j}^k) - p_h(t_j,y_j,u_{h,j}) \ge 0.$$

By Lemma 3.6 we have

$$0 \le J_h(t,x,\{u_{h,j}^k\}) - v_h(t_v,x) \le$$

$$\le \sum_{j=0}^{N-1} [\,v_h(t_{j+1},y_{j+1}) - v_h^k(t_{j+1},y_{j+1}) + \left(p_h^k(t_j,y_j,u_{h,j}^k) - p_h(t_j,y_j,u_{h,j})\right)\Big] =$$

$$= \sum_{j=0}^{N-1} [\,v_h(t_{j+1},y_{j+1}) - v_h^k(t_{j+1},y_{j+1}) + \left(v_h^k(t_{j+1},\bar{y}_{j+1}) - v_h(t_{j+1},\bar{y}_{j+1})\right)\Big] \le$$

$$\le 2\bar{C}_2 N\left(\frac{k}{h}+k\right) = 2\bar{C}_2 T(1+h)\frac{k}{h^2} \le \bar{C}_4\frac{k}{h^2},$$

where $\bar{y}_{j+1} = y_j + hf\left(t_j,y_j,+\frac{h}{2}f(t_j,y_j,u_{h,j})u_{h,j}\right)$.

Theorem 4.3. *Suppose that the assumptions (A1)-(A4) hold. Let $v(t,x)$be a value function of (2.1) and $u_h^k(\cdot)$be the feedback control (4.4). Then for any $(t,x) \in [0,T]\times\bar{\Omega}$we have*

$$\lim_{h\to 0} J\left(t,x,u_h^k(\cdot)\right) = v(t,x)$$

when $k = O(h^{2+\gamma})$, $0<\gamma<1$.

Proof. For any $(t,x) \in [0,T]\times\bar{\Omega}$we have

$$| J\left(t,x,u_h^k(\cdot)\right) - v(t,x) | \le$$

$$\le | J\left(t,x,u_h^k(\cdot)\right) - J_h^k(t,x,\{u_h^k\}) | + | J_h(t,x,\{u_h^k\}) - v_h(t,x) | + | v_h(t,x) - v(t,x) |$$

It follows from Lemma 4.1 and Lemma 4.2 that

$$| J\left(t,x,u_h^k(\cdot)\right) - v(t,x) | \le \bar{C}_3 h + \bar{C}_4\frac{k}{h^2} + | v_h(t,x) - v(t,x) |.$$

Thus, from Theorem 3.5, we complete the proof by setting $k = O(h^{2+\gamma})$, $0<\gamma<1$.

5. APPROXIMATION ALGORITHM

5.1. The Algorithm

Based on the full discretization scheme (3.11) and convex linear combination scheme (3.12), we present the algorithm as follows.

5.1.1. Algorithm for Computing Viscosity Solutions

Step 1: Define time-division point set $S = \{t_j, |, j, = 01 \dots N\}$ and space-division point set $G = \{x_i, |, i, = 12 \dots M\}$ with $h = \frac{T}{N} \in (0,1)$ and $t_j = jh(j, =, 0, 1 \dots N)$. Let $U^N = (U_1^N, \dots, U_M^N) = \big(g, (x_1), \dots g(x_M)\big)$.

Step 2: j=N

Step 3: Compute $U_i^{j-1}$of $U^{j-1} = \big(U_1^{j-1}, \dots, U_i^{j-1}\big)$as follows.

Step 3.1: Discretize the control domain $U$by $U_d = \{u_n, :, n = 12 \dots \}$with a given control-space mesh $d > 0$.

Step 3.2: For a fixed every $u \in U_d$, compute $\lambda_{ij-1}^k(u)$satisfying the following:

$$x_i + hf\left(t_{j-1}, x_i, +\frac{1}{2}hf\big(t_{j-1}, x_i, u\big)u\right) = \sum_{k=1}^{M} \lambda_{ij-1}^k(u) x_k.$$

Step 3.3: Find

$$u^*\big(t_{j-1}, x_i\big) = \arg\min_{u \in U_d}\left\{h\ell\big(t_{j-1}, x_i, u\big) + \sum_{k=1}^{M} \lambda_{ij-1}^k(u) U_k^j\right\},$$

and put

$$U_i^{j-1} = h\ell\left(t_j, x_i, u^*\big(t_{j-1}, x_i\big)\right) + \sum_{k=1}^{M} \lambda_{ij-1}^k\left(u^*\big(t_{j-1}, x_i\big)\right) U_k^j.$$

Step 4: If $0 < j$, put it to $j = j - 1$and go to the step 3, otherwise stop.

5.1.2. Algorithm for Determining the Optimal Feedback Control and the Optimal Trajectory

Step 1: Set $y_0 = x$

Step 2: Compute $u^*\big(t_j, y_j\big)$and y_{j+1}as follows:

Find the space division point $x_k \in G$close to $y_j$and then set

$$u^*\big(t_j, y_j\big) = u^*\big(t_j, x_k\big).$$

Determine y_{j+1} by

$$y_{j+1} = y_j + hf\left(t_j, y_j, +\frac{1}{2}hf\left(t_j, y_j, u^*\big(t_j, y_j\big)\right)u^*\big(t_j, y_j\big)\right).$$

Step 3: If $j < N$, put $j = j + 1$and go to the step 2, otherwise stop.

5.2. Numerical Experiments

Example 5.1. *([20, 21]) We apply the algorithm of Subsection 5.1 to solve the following optimal control problem*

$$\begin{cases} \dot{y}(t) = 2\big(1 - y(t)\big)u(t), & t \in (0,1] \\ y(0) = z \\ u(t) \in U = [0,1] \\ J\big(u(\cdot)\big) = \int_0^1 | 1 - y(t) | (1+t)^2 u(t)^2 dt + 2 | 1 - y(1) | \to \min, \end{cases} \tag{5.1}$$

where $z \in R$ is a given initial value.

$$| f(t,x,u) | = 2 | (1-x)u | \leq 2 | 1 - x | \leq 2(1 + | x |),$$

$$| \ell(t,x,u) | = | 1-x | (1+t)^2 u^2 \leq 4 | 1-x | \leq 4(1+| x |),$$
$$| g(x) | = 2 | 1-x | \leq 2(1+| x |)$$

and then for $C = 4$, the linear growth condition holds.

The HJB equation corresponding to (5.1) is

$$\begin{cases} -v_s(x,s) + \sup_{u \in U}\{-D_x v(x,s) \cdot 2(1-x)u - | 1-x | (1+s)^2 u^2\} = 0, \\ v(x,1) = 2 | 1-x |. \end{cases} \quad (5.2)$$

Therefore, $v(x,s) = | 1-x | (1+s)$is a viscosity solution of (5.2) and the optimal trajectory is $y(t) = 1$when $x = 1$. By (5.2), the optimal control is

$$u^*(s) = \frac{1}{1+s}$$

when $x \neq 1$and the optimal trajectory is

$$y^*(t) = \frac{t^2 + 2t + z}{(1+t)^2}.$$

The space mesh size must satisfy $k = Ch^{2+\gamma}, 0 < \gamma < 1$, where $C$is arbitrarily given constant. We have chosen the same time step $h = 0.01$and space step $k = 0.005$as the previous work [20] for comparison. The following figures show the value function and optimal feedback control, respectively (Figure 1).

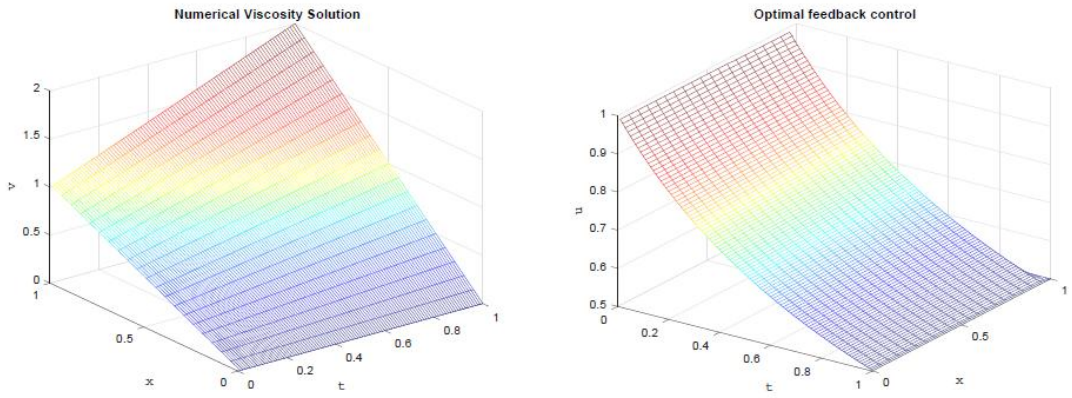


Figure 1. Numerical viscosity solution and optimal feedback control

Figures 2 and 3 show the numerical solutions of the optimal feedback control and trajectory with the initial state $z = 0.3$and $z = 0.65$, respectively.

Tables 1 and 2 show the maximum norm errors between the analytical solutions and the numerical solutions obtained by the Euler scheme and the new scheme with $h = 0.01$, respectively.

Table 1. The maximum norm errors between the numerical solutions and analytical solutions for $u^*$and $y^*$when $z = 0.3$

Scheme	Control u^*	Trajectory y^*
Euler scheme	1e-2	3.9e-3
New scheme	1e-2	1.1e-3

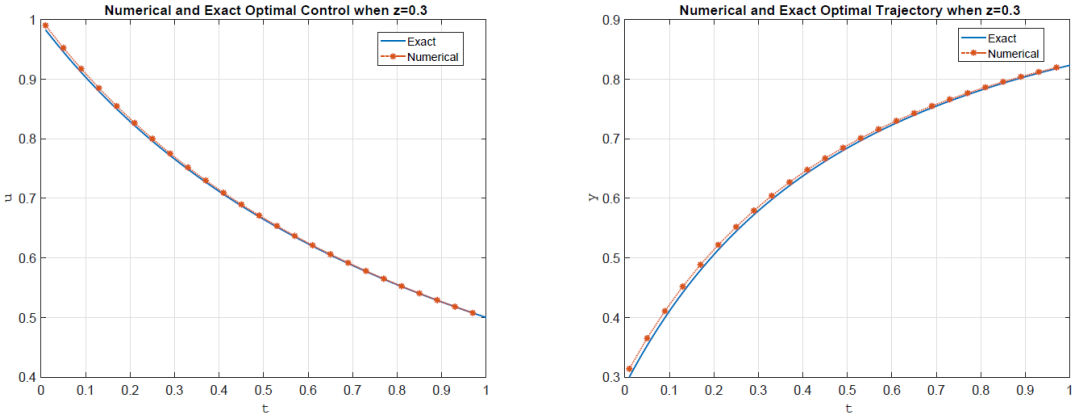


Figure 2. Numerical solutions of optimal control and trajectory when $z = 0.3$

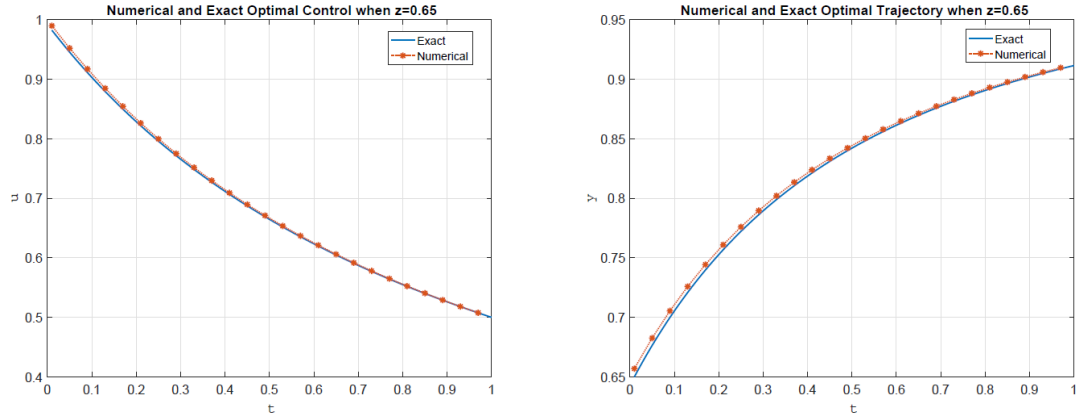


Figure 3. Numerical solutions of optimal control and trajectory when $z = 0.65$

Table 2. The maximum norm errors between the numerical solutions and analytical solutions for $u^*$and $y^*$when $z = 0.65$

Scheme	Control u^*	Trajectory y^*
Euler scheme	1e-2	1.9e-3
New scheme	1e-2	5.69e-4

Table 3 shows the maximum norm error between the analytical solution and the numerical solution obtained by the Euler scheme and new schemes at different time steps.

Table 3. The maximum norm errors between the numerical solutions and analytical solutions for $u^*$and $y^*$with $z = 0.65$at different time steps.

Scheme	$h = 0.05$		$h = 0.01$		$h = 0.005$	
	Control u^*	Trajectory y^*	Control u^*	Trajectory y^*	Control u^*	Trajectory y^*
Euler scheme	0.05	0.0104	0.01	1.9e-3	0.005	9.92e-4
New scheme	0.05	0.0032	0.01	5.69e-4	0.005	2.9e-4

Example 5.2. *Consider the following finite time optimal control problem.*

$$\begin{cases} \dot{y}(t) = 2\big(1 - y^3(t)\big)u(t), t \in (0,1] \\ y(0) = z \\ u(t) \in U = [0,1] \\ J\big(u(\cdot)\big) = \int_0^1 |1 - y(t)|\,(1+t)^2 u(t)^2 dt + 2\big(1 - y(1)\big)^2 \to \min_u \end{cases} \tag{5.3}$$

For (5.3), we have

$$| f(t,x,u) |= 2 | (1-x^3)u |\leq 2(1+| x |^3)$$
$$| \ell(t,x,u) |=| 1-x | (1+t)^2u^2 \leq 4 | 1-x |\leq 4(1+| x |) \leq$$
$$\leq 4(1+| x |)^2 = 4(1+| x |^2+ 2 | x |) \leq 8(1+| x |^2)$$
$$| g(x) |= 2 | (1-x)^2 |\leq 2(1+| x |^2)$$

Also, for $f(t,x,u)$*, we have*

$$(f,(t,x,u)x) = (2,(1-x^3)ux) \leq 2x \leq (1+| x |^2),$$

which implies that the assumption (A2) is satisfied and then for $C = 8$*and* $p = 3$*, the growth-condition is satisfied.*

The viscosity solution of the HJB equation and the optimal feedback control for the time step $h = 0.01$and the space step $k = 0.005$are as follows (Figure 4).

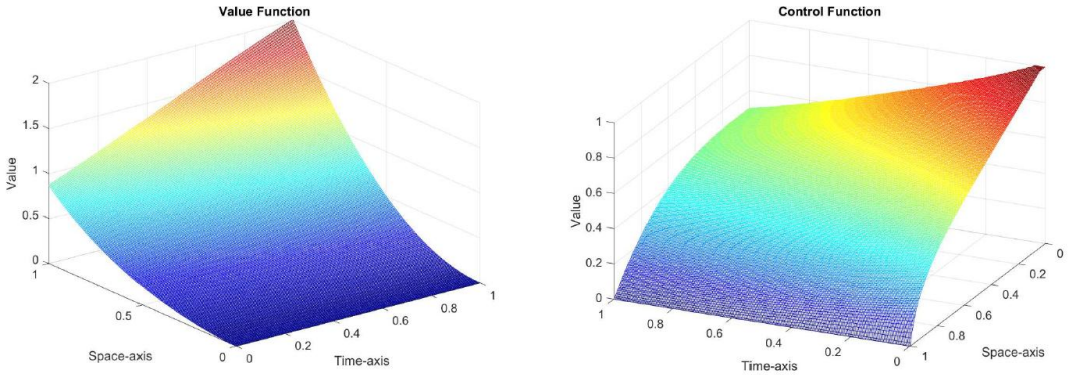


Figure 4. Numerical viscosity solution and optimal feedback control function

In Figure 5, the numerical solution of the optimal feedback control and the optimal trajectory are shown for the initial state $z = 0.65$.

Example 5.3. Consider the following finite time optimal control problem.

$$\begin{cases} \dot{y}_1(t) = y_2(t), \dot{y}_2(t) = u(t), & t \in (0,1] \\ y_1(0) = x_1, y_2(0) = x_2 \\ u(t) \in U = [0,1] \\ J\big(u(\cdot)\big) = y_1^2(1) + y_2^2(1) \rightarrow \min_u \end{cases} \tag{5.4}$$

The following figures show the value function and optimal feedback control for time-step $h = 0.005$ and space-step $k = 0.001$ (Figure 6).

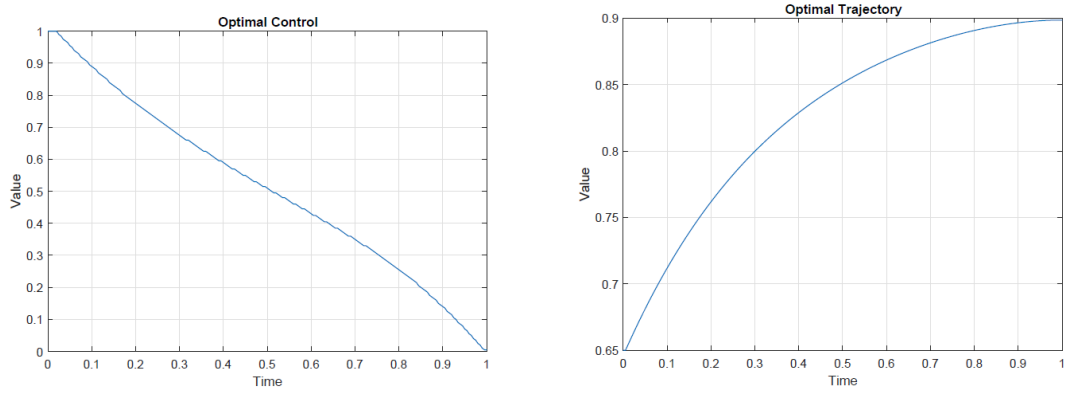


Figure 5. Numerical solution of optimal control and optimal trajectory for $z = 0.65$

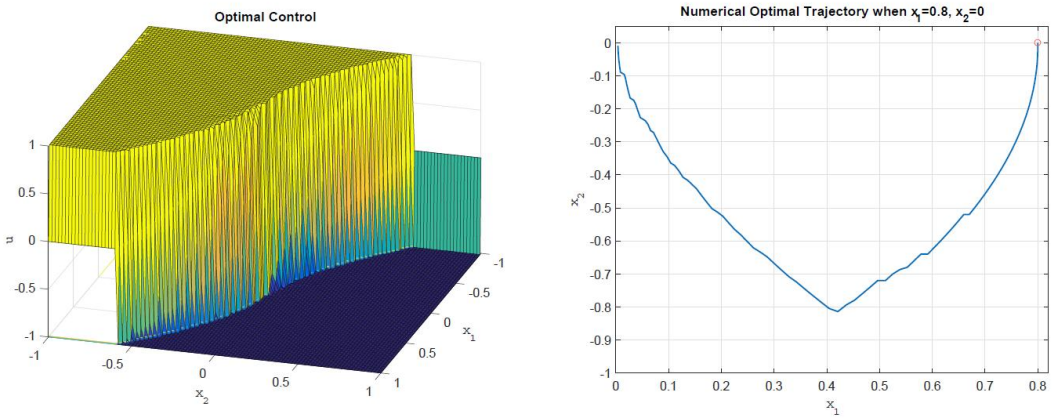


Figure 6. Numerical optimal feedback control and optimal trajectory

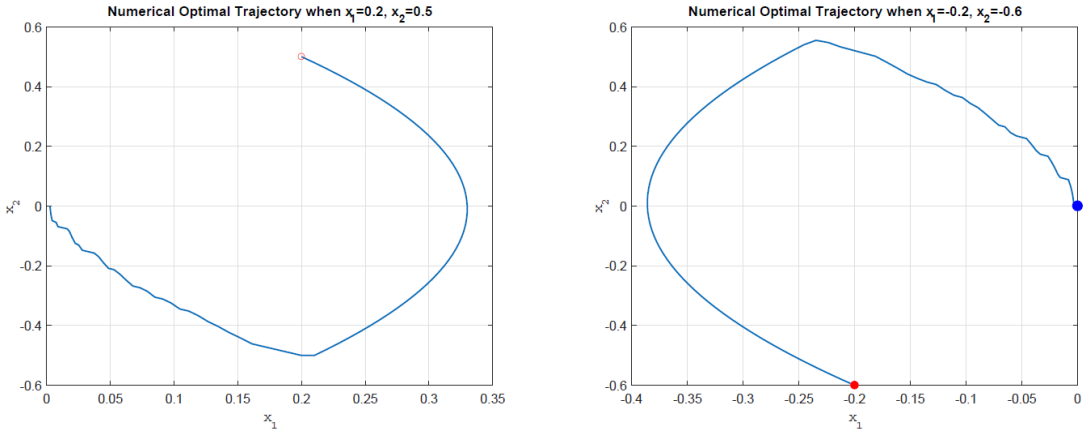


Figure 7. Numerical optimal trajectory with initial point

6. CONCLUSION

We proved the comparison principle of viscosity solution by converting unbounded function into bounded one through suitable transformation under the growth conditions for the control system and the cost functional, and proved that the value function is the unique unbounded viscosity solution of the HJB equation. The convergence of value function approximation scheme based on discrete time dynamic programming is proved, and the optimal feedback control for the control systems is constructed based on the approximations of the viscosity solution. The numerical results of the time-space discretization algorithm by the Euler implicit scheme show the efficiency of our proposed method.

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